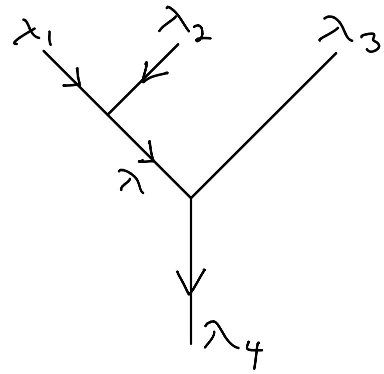
Proposition 2 $\longrightarrow$ system of fundamental solutions of KZ eq. around $u_1 = u_2 = 0$ written as

$$\Phi_1(u_1, u_2) = \varphi_1(u_1, u_2)\, u_1^{\frac{1}{K}\Omega_{12}}\, u_2^{\frac{1}{K}(\Omega_{12}+\Omega_{13}+\Omega_{23})}$$

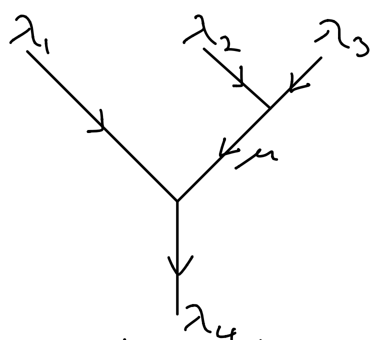
with holomorphic function $\varphi_1(u_1, u_2)$.

Φ_1 is <u>matrix-valued</u> and diagonalized with respect to

$\{p_\lambda\} =$ 

("tree basis" of conformal blocks)

Similarly, can construct horizontal sections of $\mathcal{E}$ associated to

$\{p_\mu\} =$ 

$\longrightarrow$ perform coordinate transformation

$$\zeta_2 - \zeta_1 = v_1 v_2, \quad \zeta_2 = v_2$$

$\left(v_1 v_2 = z_3 - z_2, \quad v_2 = z_3 - z_1 \Rightarrow v_1 = \frac{z_3 - z_2}{z_3 - z_1} \right)$

$\longrightarrow$ connection matrix ω is expressed around $v_1 = v_2 = 0$ as

$$\omega = \frac{1}{K} \left(\frac{\Omega^{(23)}}{v_1} dv_1 + \frac{\Omega^{(12)} + \Omega^{(13)} + \Omega^{(23)}}{v_2} dv_2 + \omega_2 \right)$$

where ω_2 is hol. 1-form around $v_1 = v_2 = 0$

$\Omega^{(23)}$ and $\Omega^{(12)} + \Omega^{(13)} + \Omega^{(23)}$ are diagonalized simultaneously with respect to $\{P_n\}$.

Solutions of KZ eq. around $v_1 = v_2 = 0$ become:

$$\Phi_2(v_1, v_2) = \varphi_2(v_1, v_2)\, v_1^{\frac{1}{K}\Omega^{(23)}}\, v_2^{\frac{1}{K}(\Omega^{(12)} + \Omega^{(13)} + \Omega^{(23)})}$$

where $\varphi_2(v_1, v_2)$ is hol. around $v_1 = v_2 = 0$.

Note that $u_2 = z_3 - z_1$, $u_1 = \frac{z_2 - z_1}{z_3 - z_1}$ and

$$v_2 = z_3 - z_1, \quad v_1 = \frac{z_3 - z_2}{z_3 - z_1}$$

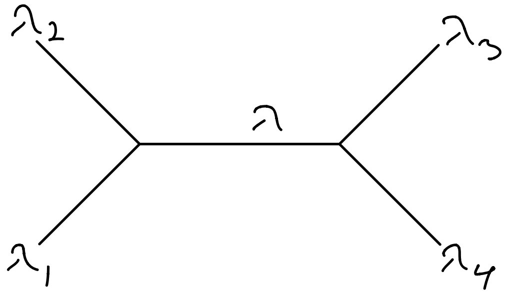
$\Rightarrow$ Φ_1 corresponds to asymptotic region

(A) $z_1 < z_2 \ll z_3$ and Φ_2 to the region

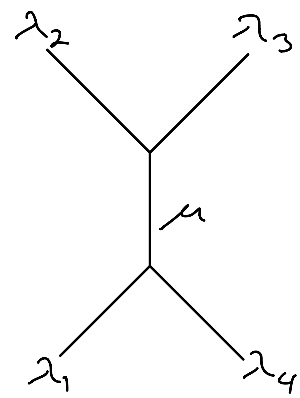
(B) $z_1 \ll z_2 < z_3$

$\longrightarrow$ Analytic continuation from region (A) to region (B) gives

$\Phi_1 = \Phi_2 F$, where F is called connection matrix.



$$= \sum_\mu F_{\mu\lambda}$$



$(((12)3)4)$ $\qquad$ $((1(23))4)$

In terms of chiral vertex operators this gives

<u>Lemma 3:</u>

In the region $0 < |\zeta_1| < |\zeta_2|$ we have

$$\Psi^{\lambda_4}_{\lambda\lambda_3}(\zeta_2)\left(\Psi^{\lambda}_{\lambda_1\lambda_2}(\zeta_1)\otimes id_{H_{\lambda_3}}\right)$$
$$= \sum_\mu F_{\mu\lambda}\Psi^{\lambda_4}_{\lambda_1\mu}(\zeta_2)\left(id_{H_{\lambda_1}}\otimes\Psi^{\mu}_{\lambda_2\lambda_3}(\zeta_2-\zeta_1)\right)$$

Φ_1 and Φ_2 can be understood as solutions of Fuchsian differential equation

$$G'(x) = \frac{1}{k}\left(\frac{\Omega^{(12)}}{x} + \frac{\Omega^{(23)}}{x-1}\right)G(x)$$

with regular singularities at $0, 1$ and ∞:

$$\Phi(z_1, z_2, z_3) = (z_3 - z_1)^{\frac{1}{k}\left(\Omega^{(12)}+\Omega^{(13)}+\Omega^{(23)}\right)}\, G\left(\frac{z_2-z_1}{z_3-z_1}\right)$$

Φ_1 and Φ_2 then correspond to the two solutions:

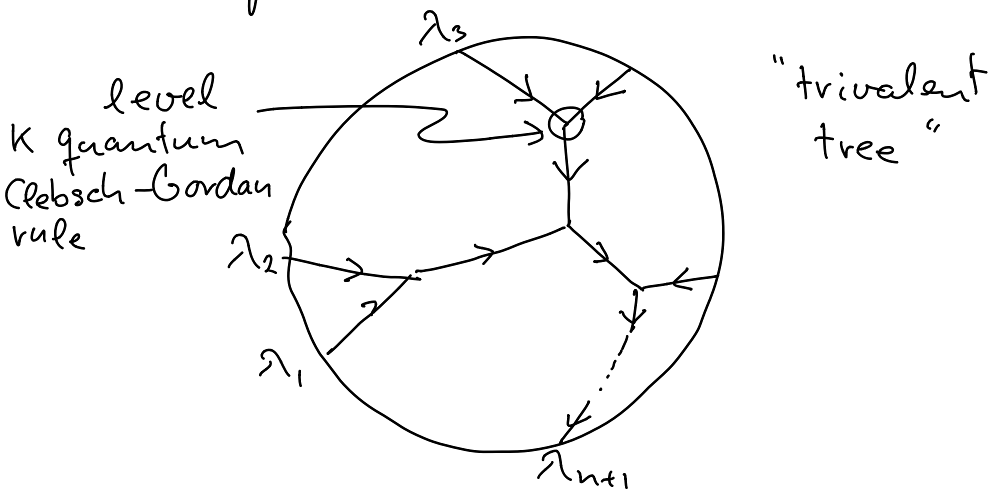
$$G_1(x) = H_1(x)\, x^{\frac{1}{k}\Omega_{12}} \quad (x \to 0), \quad H_1 \text{ hol.}$$

$$G_2(x) = H_2(x)\,(1-x)^{\frac{1}{k}\Omega_{23}}$$

around $x=0$ and $x=1$. We have

$$G_1(x) = G_2(x)\, F$$

Next, let T_n be a trivalent tree with $n+1$ external edges:



Take $n+1$ points $p_1, \dots, p_n, p_{n+1}$ on the Riemann sphere with $p_{n+1} = \infty$ and set $z(p_j) = z_j$

$\longrightarrow$ space of conformal blocks:

$$\mathcal{H}(p_1, \dots, p_n, p_{n+1}; \lambda_1, \dots, \lambda_n, \lambda_{n+1}^*)$$

with basis given by labellings of above tree.

Each trivalent tree represents a system of solutions of the KZ equation:

Consider tree of type $(\cdots(12)3)\cdots n)n{+}1)$

$\longrightarrow$ perform coordinate transformations
$\zeta_k = z_{k+1} - z_1$ and $\zeta_k = u_k u_{k+1} \cdots u_{n-1}$,
$k = 1, \ldots, n-1$

$\longrightarrow$ have solutions of KZ equations around $u_1 = \cdots = u_{n-1} = 0$:

$$\Phi_1 = \varphi_1(u_1, \ldots, u_{n-1})\, u_1^{\frac{1}{k}\Omega^{(12)}}\, u_2^{\frac{1}{k}\left(\Omega^{(12)} + \Omega^{(13)} + \Omega^{(23)}\right)} \cdots u_{n-1}^{\frac{1}{k}\sum_{1 \leq i < j \leq n} \Omega^{(ij)}}$$

where $\varphi_1(u_1, \ldots, u_{n-1})$ is matrix-valued hol. function.

$\longrightarrow$ Φ is diagonalized with respect to basis corresponding to $(\cdots(12)3)\cdots n)n{+}1)$

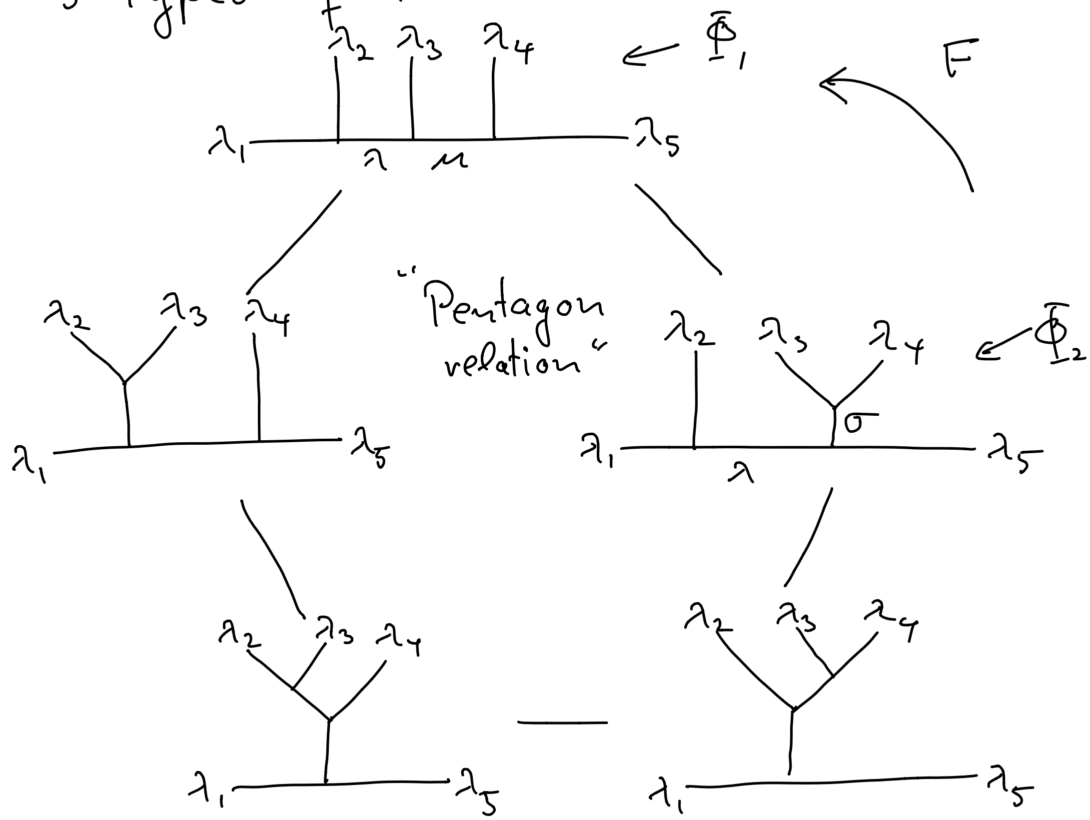
<u>Consider the case $n=4$:</u>

For the tree of type $(((12)(34))5)$ perform coordinate transformation $\zeta_1 = v_1 v_3$, $\zeta_3 - \zeta_2 = v_2 v_3$, $\zeta_3 = v_3$ and associate

$$\Phi_2 = \varphi_2(v_1, v_2, v_3)\, v_1^{\frac{1}{k}\Omega^{(12)}}\, v_2^{\frac{1}{k}\Omega^{(34)}}\, v_3^{\frac{1}{k}\sum_{1 \leq i < j \leq 4} \Omega^{(ij)}}$$

where $\varphi_2(v_1, v_2, v_3)$ is holomorphic around $v_1 = v_2 = v_3$.

5 types of trees:



One can go from graph $((((12)3)4)5)$ to graph $(((12)(34))5)$ by the connection matrix of Lemma 3 as follows:

$$(*) \quad \Psi^{\lambda_5}_{\mu\lambda_4}(\zeta_2)\left(\Psi^{\mu}_{\lambda\lambda_3}(\zeta_1) \otimes id_{H_{\lambda_4}}\right)$$

$$= \sum_{\sigma} F_{\sigma\mu} \Psi^{\lambda_5}_{\lambda\sigma}(\zeta_2)\left(id_{H_\lambda} \otimes \Psi^{\sigma}_{\lambda_3\lambda_4}(\zeta_2 - \zeta_1)\right)$$

In general, the connection matrix for each edge of the Pentagon above is represented by the composition of connection matrices for $n=3$. $\longrightarrow$ call $n=3$ connection matrix "elementary c.m." and the linear map (*) is called "elementary fusion operation"

We also have (without proof)

<u>Proposition 3:</u>

For the two edge paths connecting two distinct trees in the Pentagon diagram, the corresponding compositions of elementary connection matrices coincide.

<u>Monodromy representation of braid group:</u>

Take n distinct points $p_1, p_2, \ldots, p_n$ with coordinates satisfying $0 < z_1 < z_2 < \cdots < z_n$

$\longrightarrow$ associate level k highest weights $\lambda_1, \ldots, \lambda_n$ to $p_1, \ldots, p_n$. Take $p_0 = 0$, $p_{n+1} = \infty$ with $\lambda_0 = 0$ and $\lambda_{n+1} = 0$.

$\longrightarrow$ corresponding conformal blocks:

$$\mathcal{H}(p_0, p_1, \ldots, p_{n+1}; \lambda_0, \lambda_1, \ldots, \lambda_{n+1}^*)$$

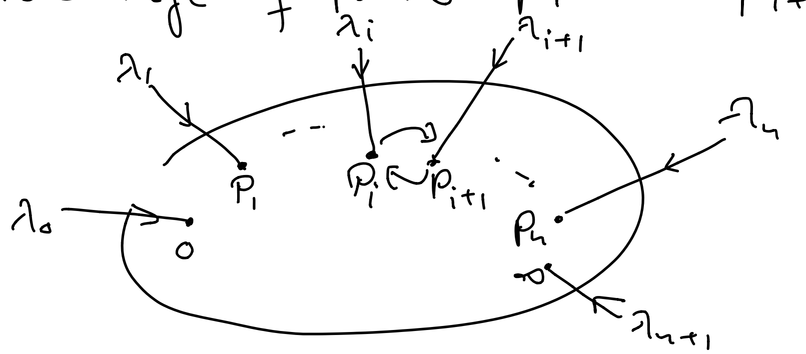
$$\hookrightarrow \mathrm{Hom}_{\mathfrak{g}}\left(\bigotimes_{j=1}^{n} V_{\lambda_j}, \mathbb{C}\right)$$

Denote image of above map by $V_{\lambda_1,\dots,\lambda_n}$ with basis given by $\{v_{\mu_0\mu_1\dots\mu_n}\}$ such that any triple $(\mu_{j-1}, \lambda_j, \mu_j)$ satisfies quantum Clebsch-Gordan condition at level K.

$\longrightarrow$ a generator σ_i of the braid group B_n defines a linear map:

$$\rho(\sigma_i) : V_{\lambda_1 \dots \lambda_i \lambda_{i+1} \dots \lambda_n} \longrightarrow V_{\lambda_1 \dots \lambda_{i+1} \lambda_i \dots \lambda_n}$$

"interchange of points p_i and p_{i+1}":



$\rho(\sigma_i)$ only depends on homotopy class of above path as KZ connection is flat.

In general we have for $\sigma \in B_n$:

$$\rho(\sigma) : V_{\lambda_1 \dots \lambda_n} \longrightarrow V_{\lambda_{\pi\circ\sigma(1)} \dots \lambda_{\pi\circ\sigma(n)}}$$

where $\pi : B_n \longrightarrow S_n$ is natural surjection.

We also have: $\rho(\sigma\tau) = \rho(\sigma)\rho(\tau)$, $\sigma, \tau \in B_n$

Let us deal with the case $n=3$:

For the solution

$$\Phi_1(u_1, u_2) = \varphi_1(u_1, u_2)\, u_1^{\frac{1}{k}\Omega_{12}}\, u_2^{\frac{1}{k}(\Omega_{12}+\Omega_{13}+\Omega_{23})}$$

with $u_2 = z_3 - z_1$, $u_1 = \dfrac{z_2 - z_1}{z_3 - z_1}$, we see

$$\rho(\sigma_1)\, u_1 = -u_1 \Rightarrow \rho(\sigma_1)\, \Phi_1 = P_{12} \exp\left(\frac{\pi\sqrt{-1}}{k}\Omega_{12}\right)\Phi_1$$

where $P_{12} : V_{\lambda_1 \lambda_2 \lambda_3} \longrightarrow V_{\lambda_2 \lambda_1 \lambda_3}$

Ω_{12}/k is diagonalized for tree basis $(((12)3)4)$ with eigenvalue $\Delta_\lambda - \Delta_{\lambda_1} - \Delta_{\lambda_2}$

On the other hand:

$$\rho(\sigma_2)\, \Phi_2 = P_{23} \exp\left(\pi\sqrt{-1}\,\Omega_{23}/k\right)\Phi_2,$$

where Ω_{23}/k is diagonalized with respect to basis $((1(23))4)$.

$\longrightarrow$ To combine these <u>local</u> monodromies, we need connection matrix F:

$$\Phi_1 = \Phi_2 F$$